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1991 J. Phys. A: Math. Gen. 24 2287

(http://iopscience.iop.org/0305-4470/24/10/017)

J. Phys. A: Math. Gen. 24 (1991) 2287–2314. Printed in the UK

On the existence of infinite period-doubling sequences in a class of 4D semi-symplectic mappings

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Received 14 June 1990, in final form 31 January 1991

Abstract. We investigate the existence of an infinite period-doubling sequence in the following class of semi-symplectic maps: a two-dimensional (2D) nonlinear constant Jacobian map with an infinite period-doubling sequence, linearly and weakly coupled with a 2D linear map. We introduce the property of semi-symplecticity as the generalization of symplecticity that incorporates uniform dissipation. We define Krein signatures for pairs of complex eigenvalues and show that they play the same role as in the symplectic case. The Krein signature alternates in the period-doubling sequence of a 2D constant Jacobian map, whereas the signatures of iterates of a 2D linear map form (almost always) an uncorrelated row. With these results we show that any finite coupling strength destroys the infinite period-doubling sequence of the conservative maps of our class in two ways: firstly, there are 'bubbles of instability'; secondly, and far more importantly, there are period-doubling bifurcations in which the newborn period-doubled orbits are unstable. So crucial parts of the period-doubling sequence are experimentally invisible. For the dissipative maps of our class the same conclusions hold, but only if the coupling strength is strong enough.

1. Introduction

Period-doubling sequences seem to occur in many discrete dynamical systems. They are well understood for 1D unimodal maps and for constant Jacobian maps in two dimensions. However, for higher-dimensional maps the situation is not at all clear.

In this paper we investigate the existence of period-doubling sequences in one of the simplest higher-dimensional cases: a 2D nonlinear constant Jacobian map with a period-doubling sequence, linearly and weakly coupled with a 2D linear map. If there is no coupling, this 4D map clearly has an infinite period-coupling sequence. Our main question is: does the period-doubling sequence, which exists in the uncoupled case, survive the coupling?

We require that the combined system is what we call 'semi-symplectic'. Such systems have a constant Jacobian and can be seen as a high-dimensional generalization of the constant Jacobian maps in two dimensions. The semi-symplectic maps reduce to symplectic ones in the case of unit Jacobian. Their multipliers have symmetry properties with respect to a symmetry circle that are similar to those of the symplectic maps. These properties impose severe restrictions on the possible spectra of the derivative matrices. This gives us a maximum of control in creating high-dimensional period-doubling sequences.

Symplectic maps are interesting because they arise as stroboscopic maps of periodically driven Hamiltonian systems. A Hamiltonian system with n degrees of freedom creates in this way a $2n$-dimensional symplectic map. Similarly, semi-symplectic maps arise if to the Hamiltonian flow friction linearly proportional to the momenta is added (Valkering 1988). So our investigated 4D mapping can be thought of as the stroboscopic map of two periodically driven linearly coupled oscillators (one anharmonic, one harmonic) damped by the same friction.

The period-doubling sequence in the uncoupled case is easy to understand. Obviously the additional linear map does not influence the dynamics of the 2D nonlinear map. At one stage of the period-doubling sequence the eigenvalues of the linearization round a fixed point behave as in figures 1 or 2. Note that all the eigenvalues are on the symmetry circle and must meet somewhere. When we introduce some weak coupling, the positions of the eigenvalues can change only slightly. Moreover, we will see that, because of the semi-symplecticity, simple eigenvalues on the symmetry circle cannot leave this circle due to a small perturbation. So a periodic orbit can only lose its stability after collisions of eigenvalues on the symmetry circle. These collisions can occur on the real axis (at least two eigenvalues are involved) or somewhere in the complex plane on the symmetry circle (four eigenvalues are involved).

Whether after collisions in the second case the eigenvalues actually move off the symmetry circle, or not, depends on the Krein signatures of these eigenvalues. The Krein signature is an important property of a complex pair of eigenvalues on the symmetry circle of a semi-symplectic map. The definition and results on the Krein signature we present are simple generalizations of this concept in the symplectic (conservative) case (Howard and MacKay 1987). As in the well-known conservative case, a collision of eigenvalues on the symmetry circle can only lead to eigenvalues

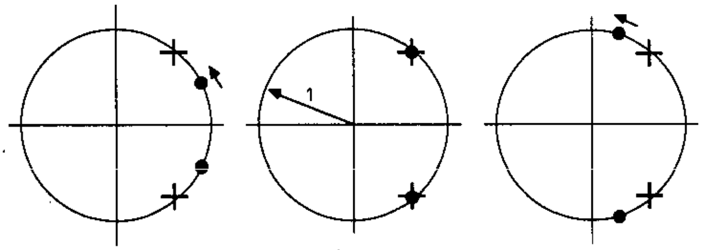

Figure 1. Behaviour of eigenvalues in one part of the period-doubling sequence of a symplectic map (the conservative case) with no coupling. The eigenvalues simply pass one another on the unit circle.

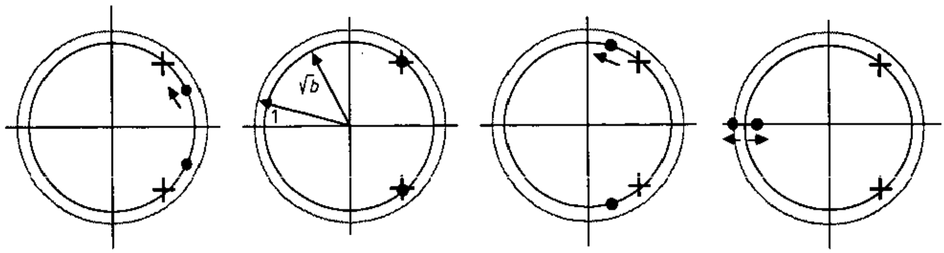

Figure 2. Behaviour of eigenvalues in one part of the period-doubling sequence of a semi-symplectic map (the dissipative case) with no coupling. The eigenvalues simply pass one another on the symmetry circle.

splitting off the symmetry circle if these eigenvalues have opposite Krein signatures. In this way a periodic orbit may lose its stability. In order to say something about the existence of a whole period-doubling sequence in the weakly coupled case, we have to investigate at every stage of the period-doubling sequence the possible occurrence of eigenvalues with opposite signature. Moreover, we have to determine the consequences of collisions of such eigenvalues.

The set-up of the paper is as follows.

In section 2 we give the definition of semi-symplectic mappings and discuss their general properties.

In section 3 we describe the class of studied maps and question the existence of an infinite period-doubling sequence.

In section 4, Krein signatures for eigenvalues of semi-symplectic maps are defined and discussed. Some known results are formulated as apply to our problem.

In section 5, Krein signatures for the period-doubling sequence in the uncoupled case are investigated. The main result is that the signatures of the 2D period-doubling map alternate in a period-doubling sequence, whereas those of the linear map form a uncorrelated sequence (for almost all parameter values). Collisions of opposite-signed eigenvalues in the period-doubling sequence are inevitable and also must occur very close to the real axis.

In section 6 the changes in the spectrum of the 4D map, when changing the bifurcation parameter, are analysed. The relevant unfoldings of multiple eigenvalues are calculated explicitly for the period one case. Collision of eigenvalues with opposite signatures gives rise to a 'bubble of instability', whereas equal signatures generate an 'avoided crossing'. We argue that the higher-periodic cases are typically the same, with some properly scaled parameters.

In section 7 it is shown that this implies that for any finite coupling in the conservative case there are small 'bubbles of instability' in the period-doubling sequence. Moreover, and more importantly, there must (almost always) be period-doubling bifurcations in which the newborn periodic orbit is initially unstable. We can no longer speak of a period-doubling sequence in the conventional way.

In the dissipative case the same conclusion holds if the coupling strength is above some threshold. Moreover, whenever the stability in the period-doubling sequence is lost and we change the bifurcation parameter adiabatically, we may miss the rest of the period-doubling sequence due to hysteresis effects.

2. Semi-symplectic maps

In this section we define semi-symplectic matrices and maps and we describe some of their elementary properties.

2.1. Definitions of semi-symplectic matrices and maps

Let J be a $2n\times 2n$ real matrix obeying $J^2=-\mathbb{1}$ and $J^{\mathrm{T}}=-J$.

Definition. We call a $2n\times 2n$ real matrix M semi-symplectic (with respect to J, b) if there exists a $b>0$ such that

$$M^{\mathrm{T}}JM=bJ \qquad \det(M)>0. \tag{2.1}$$

Consider a mapping $F:\mathscr{D}\to\mathscr{D}$ where $\mathscr{D}\subset\mathbb{R}^{2n}$. Our restriction to Euclidean spaces is just for convenience and by no means crucial. Denote the derivative matrix of F by DF.

Definition. We call a map $F(x)$ semi-symplectic if there exists one constant $b>0$ and one matrix J such that $DF(x)$ is semi-symplectic with respect to J, b for all $x\in\mathscr{D}$.

F is symplectic (Arnold 1978) iff F is semi-symplectic and $b=1$. Note that the definition of semi-symplectic maps implies that b does not depend on x and that

$$\det(DF(x))=b^n \qquad \forall x\in\mathscr{D}. \tag{2.2}$$

Moreover, in two dimensions ($n=1$) semi-symplecticity is equivalent to a constant positive Jacobian (as can be readily verified).

2.2. *Elementary properties*

The semi-symplectic maps and matrices possess nice properties equivalent to the well-known ones for the symplectic maps and matrices (Arnold 1978). We list the properties we need.

Let M be any semi-symplectic matrix. From the definition of semi-symplectic matrices it follows directly that

$$(Mx, JMy)=b(x, Jy) \qquad \forall x, y\in\mathscr{D}. \tag{2.3}$$

One can derive straightforwardly that the product of semi-symplectic matrices and the composition of semi-symplectic maps are again semi-symplectic. Also, the inverse of a semi-symplectic map is again semi-symplectic, and the identity map is semi-symplectic too. So the semi-symplectic maps form a group.

Eigenvalues of semi-symplectic matrices have symmetry properties similar to those of symplectic ones. Those are, in short, that the spectrum of a symplectic matrix must be symmetric with respect to the real axis and with respect to the unit circle (Arnold 1978). The same property holds for semi-symplectic matrices, if the unit circle is replaced by a circle with radius $\sqrt{b}$ (figure 3). From here on we call this circle the symmetry circle. That the symmetry is there follows immediately from the fact that if

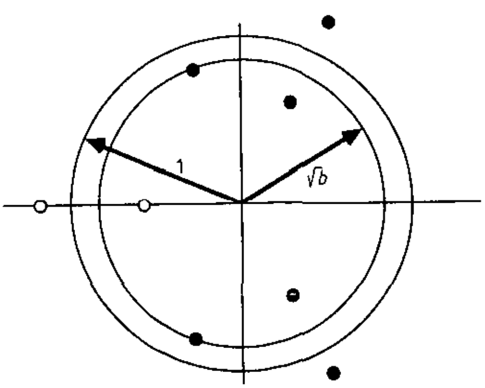

Figure 3. Possible positions of the eigenvalues of a semi-symplectic map with dissipation parameter b: ●, quadruplet; ◉, complex pair; ○, real pair.

M is semi-symplectic, then $M/\sqrt{b}$ is symplectic (cf equation (2.1)). It can also be derived explicitly from the characteristic polynomial $S(\lambda) \equiv \det(M - \lambda \mathbb{1})$, which has the nice property

$$S\left(\frac{b}{\lambda}\right) = \frac{b^n}{\lambda^{2n}} S(\lambda). \tag{2.4}$$

For by the semi-symplecticity of M (2.1):

$$\begin{aligned} S(b/\lambda) &= \det(M - b/\lambda \mathbb{1}) \\ &= (1/\lambda)^{2n}[\det(\lambda JM - bJ)]/\det(J) \\ &= (1/\lambda)^{2n}[\det(\lambda JM - M^{\mathrm{T}}JM)]/\det(J) \\ &= (1/\lambda)^{2n} \det(\lambda \mathbb{1} - M^{\mathrm{T}}) \det(M) = (b/\lambda^2)^n S(\lambda). \end{aligned}$$

So, if λ is an eigenvalue of a semi-symplectic matrix M, then also b/λ, λ^* and b/λ^* must be eigenvalues (though not necessarily all different) and with the same multiplicity. There are three ways to satisfy this condition (figure 3). Every eigenvalue must be a member of: (i) a quadruplet (λ, λ^*, b/λ and b/λ^* all different); (ii) a real pair ($\lambda = \lambda^*$, $b/\lambda = b/\lambda^*$); or (iii) a complex pair with absolute value $\sqrt{b}$ ($\lambda = b/\lambda^* \neq \lambda^*$, λ^*), thus on the symmetry circle.

An eigenvalue $\lambda = \pm\sqrt{b}$ is a special case of a real pair and thus must have an even multiplicity. These properties clearly restrict the possible spectra of semi-symplectic matrices significantly.

As well known (Broucke 1969), we may use these properties to reduce the characteristic polynomial. Using the symmetry (2.4) we can reduce the characteristic polynomial of degree $2n$ in λ into one of degree n in the new variable (the so-called stability index)

$$\rho = \lambda + b/\lambda. \tag{2.5}$$

There exists a straightforward connection between ρ and the three types of eigenvalues of semi-symplectic matrices. The stability index ρ is complex iff the corresponding eigenvalues λ form a quadruplet. The stability index ρ is real iff the corresponding eigenvalues λ form a (real or complex) pair. This pair is real if $|\rho| \geq 2\sqrt{b}$ and is complex if $|\rho| < 2\sqrt{b}$. These properties follow immediately, indeed, because of definition (2.5):

$$\lambda^2 - \rho\lambda + b = 0.$$

3. Investigated maps and problem

We can now formulate our problem more accurately. First we specify the class of maps which we investigate. Second we describe the period-doubling sequence of these kind of maps in the simple (decoupled) case and question its existence in the general (weakly coupled) case.

3.1. *Class of studied maps*

Throughout we consider a semi-symplectic map $F: \mathcal{D} \to \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}^4$. F is built up from a nonlinear 2D semi-symplectic map $H_c(y)$, linearly coupled with a linear map

$L_d z$. The parameters c and d govern trace(DH_c) and trace(L_d) respectively. The coupling strength is ruled by the parameter k:

$$F\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} H_c(y) \\ L_d z \end{bmatrix} + k\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \tag{3.1}$$

where $y, z \in \mathbb{R}^2$. We impose the following properties on F and its components:

(1) F is semi-symplectic with respect to J_4, b (for all c, d and k) where: (i) J_4 and b do not depend on the parameters c, d and k; (ii) J_4 is a block-diagonal sum of J_2,

$$J_4 = \begin{bmatrix} J_2 & 0 \\ 0 & J_2 \end{bmatrix}$$

where necessarily $J_2^T = -J_2$ and $J_2^2 = -\mathbb{1}$.

(2) $H_c(y)$ possesses a proper infinite period-doubling sequence in c. By proper we mean that: (i) at every period-doubling bifurcation precisely one period-doubled orbit is born out of the original; (ii) trace $DH_c^{2^q}(y^*(c))$ depends strictly monotonically on the bifurcation parameter c (where $y^*(c)$ denotes a stable 2^q periodic point); (iii) whenever $DH_c^{2^q}(y^*(c))$ possesses a double eigenvalue (at $\pm(\sqrt{b})^{2^q}$), it is not semisimple.

(3) The eigenvalues of L_d are complex and trace(L_d) depends strictly monotonically on d.

Property (1) must hold for all values of k, thus also for $k=0$. This implies that both DH_c and L_d must be semi-symplectic with respect to J_2, b. In order to get property (1) for $k \neq 0$, one must choose the coupling matrices M and N properly.

Property (3) restricts the considered values for parameter d to the interval

$$\{d \mid -2\sqrt{b} < \mathrm{trace}(L_d) < 2\sqrt{b}\}.$$

3.2. Period-doubling in the decoupled case $k=0$

If $k=0$, F is decoupled and the y-plane is an invariant stable plane. So all the (asymptotic) dynamics of F are just those of H_c. Especially, F possesses the same period-doubling sequence as H_c in c. In each step in this period-doubling sequence two eigenvalues, associated with L_d, are fixed somewhere on the $b^{q/2}$ circle in the complex plane. The two others, associated with H_c, move monotonically around as in figure 2. The two different eigenvalues thus meet precisely once somewhere on the $b^{q/2}$ circle each step (period q) in the period-doubling sequence (figures 1 or 2).

3.3. Small coupling: does the period-doubling sequence survive?

Our main interest will be in the influence of a small coupling k on the above-described period-doubling sequence in the parameter c. The position of a periodic orbit and the eigenvalues of its Jacobi matrix depends continuously on the parameters. A single period point with simple eigenvalues cannot change character (quadruplet, real pair, complex pair) due to the semi-symplecticity of the map. So great changes in the spectrum due to small perturbations may only occur at non-simple eigenvalues.

If there is no coupling ($k=0$), collisions of eigenvalues may occur at $\pm\sqrt{b}$ or somewhere on the symmetry circle. Thus also for small but non-zero coupling collisions of eigenvalues may occur at those places. In the first case a real pair may become a complex pair (or vice versa), in the second case a quadruplet may be formed out of two complex pairs (or vice versa) (figure 8). However, the second case does not typically lead to the creation (or annihilation) of a quadruplet. Whether or not a quadruplet can be created depends on the Krein signatures. This is well known for the symplectic case, but holds just the same for the semi-symplectic one, as we will show.

4. Krein signatures

We now define the Krein signatures for pairs of eigenvalues on the symmetry circle of a semi-symplectic matrix. Our definition is a straightforward generalization of the concept of the Krein signature for symplectic matrices (Arnold and Avez 1968). Moreover, we show that the Krein signature governs the behaviour of multiple eigenvalues under perturbation, in the same way as in the symplectic case. We do so by reformulating our semi-symplectic lemmas on Krein signatures in equivalent symplectic ones. Those symplectic lemmas are known to hold (Moser 1958, Krein 1955, Yakubovich and Starzhinskii 1975, Howard and MacKay 1987).

Definition 4.1 (Yakubovich and Starzhinskii 1975). Let M be a semi-symplectic matrix (with respect to J, b) with a pair of complex eigenvalues λ, $\lambda^* = b/\lambda \neq \lambda$ (thus on the symmetry circle) with generalized eigenspaces V_λ and V_λ^*. Let W_λ be the real M-invariant plane

$$W_\lambda = \{v + v^* \mid v \in V_\lambda\}.$$

Then we call the pair of eigenvalues (λ, λ^*): (i) definite, if the quantity (x, JMx) has the same sign $\forall x \in W_\lambda$, $x \neq 0$; (ii) mixed, otherwise.

Definition 4.2. If λ, λ^* is a pair of definite eigenvalues of a semi-symplectic matrix M, then we define

$$\text{Krein signature}(\lambda, \lambda^*) = \text{sign}(x, JMx) \qquad \text{for any } x \in W_\lambda, x \neq 0.$$

Now we recall some facts about Krein signatures of symplectic matrices. These facts also do hold in the semi-symplectic case. Details and proofs can be found in appendix 1.

Lemma 4.1. If λ, $\lambda^* = b/\lambda$ is a pair of simple complex eigenvalues (thus on the symmetry circle), then the pair λ, λ^* is definite.

Lemma 4.2. The Krein signature of a pair of simple complex eigenvalues of a semi-symplectic matrix cannot change due to continuous perturbations, as long as this pair remains simple and complex.

We remark that we have deliberately not considered the case $\lambda = \pm\sqrt{b}$, because it is different. However, one can define a kind of signature too with an interesting property.

Definition 4.3. Let $\lambda = \pm\sqrt{b}$ be an eigenvalue of a semi-symplectic matrix M with generalized eigenspace W_λ. Then we call λ definite if the quantity (x, JMx) is semi-definite on W_λ and not identically zero. Moreover we define in that case

$$\text{Krein signature}(\lambda) = \text{sign}(x, JMx) \qquad \text{for any } x \in W_\lambda \text{ such that } (x, JMx) \neq 0.$$

By the spectral properties (section 2) an eigenvalue $\lambda = \pm\sqrt{b}$ must have an even multiplicity. So the simplest cases are double semisimple eigenvalues $\lambda = \pm\sqrt{b}$ and double non-semisimple ones. The semisimple type is not definite; the non-semisimple is given in lemma 4.3.

Lemma 4.3. Any double non-semisimple eigenvalue $\lambda = \pm\sqrt{b}$ is definite. Moreover, any pair of simple complex eigenvalues created by a continuous perturbation must have the same signature.

The notion of the Krein signature is important because it describes what can happen with a pair of complex eigenvalues on the symmetry circle of a semi-symplectic matrix $M(c)$, when the parameter c is changed.

Consider a smooth family $M(c)$ of semi-symplectic matrices with respect to J, b (where b does not depend on c). Let $M(c_0)$ have a pair of simple complex eigenvalues. Then for c close enough to c_0, $M(c)$ must also have a pair of simple complex eigenvalues, due to the spectral properties of M (section 2). And by lemma 4.1 such a pair is definite and by continuity (lemma 4.2) its signature cannot change. The only way a simple eigenvalue can leave the symmetry circle is by collision with another eigenvalue.

Changing parameter c further, eigenvalues collide, say for $c = c_1$. One complex pair can collide at $\pm\sqrt{b}$ and become a real pair. Two complex pairs can collide somewhere on the symmetry circle and may form a quadruplet. There are two types of collisions of two complex pairs:

(1) Two complex pairs that are definite and have equal signature, colliding into a pair of multiple eigenvalues on the symmetry circle for $c = c_1$. This multiple pair is again definite, with the same signature as those from the original eigenvalues. A further change of c cannot create a quadruplet, so all eigenvalues must stay on the symmetry circle and must remain definite.

(2) Two complex pairs colliding into a pair of multiple eigenvalues other than in case (1). Primarily collision of two simple pairs with opposite signature. This multiple pair is mixed. A further change of c can create a quadruplet, so the eigenvalues may leave the symmetry circle.

The first part of both statements follows from the continuity property for collided eigenvalues. This is known in the symplectic case (Yakubovich and Starzhinskii 1975) and can be extended to the case $b < 1$ (argument given below).

Let $\lambda(c)$ and $\mu(c)$ and conjugates be two different complex pairs of eigenvalues on the symmetry circle for $c \in (c_0, c_1)$, which collide for $c = c_1$ ($\lambda(c_1) = \mu(c_1)$). Then $\lambda(c_1) = \mu(c_1)$ is definite if $\lambda(c)$ and $\mu(c)$ are definite and have equal signature for $c \in (c_0, c_1)$.

So for collisions of pairs of simple complex eigenvalues $\lambda(c)$, $\mu(c)$ and conjugate at $c = c_1$, we see that the pair of collided eigenvalues is definite (mixed) if $\lambda(c)$ and $\mu(c)$ have equal (opposite) signatures for c near c_1.

Once we have a pair of multiple eigenvalues on the symmetry circle, its response to small perturbations is ruled by the semi-symplectic versions of Krein's and Moser's theorems (Krein 1955, Moser 1958).

4.1. *Krein's theorem*

Let $M(c_1)$ be a semi-symplectic matrix with respect to J, b. Let $\lambda_1(c_1) = \lambda_2(c_1) = \ldots = \lambda_m(c_1)$, $\lambda^*(c_1) = b/\lambda(c_1) \neq \lambda(c_1)$ be a pair of multiple complex eigenvalues on the symmetry circle that is definite.

Then for all smooth families $M(c)$ of semi-symplectic matrices with respect to J, b (where b does not depend on c) for all c in a neighbourhood of c_1 the eigenvalues $\lambda_j(c)$, $\lambda_j^*(c)$ ($j = 1 \ldots m$) must be on the symmetry circle and must be definite.

4.2. Moser's theorem

Let $M(c_1)$ be a semi-symplectic matrix with respect to J, b. Let $\lambda_1(c_1)=\lambda_2(c_1)=\ldots=\lambda_m(c_1)$, $\lambda^*(c_1)=b/\lambda(c_1)\neq\lambda(c_1)$ be a pair of multiple complex eigenvalues on the symmetry circle that is mixed.

Then there exist a smooth family $M(c)$ semi-symplectic with respect to J, b (where b does not depend on c), such that for all $c_1\leq c<c_2$ four eigenvalues $\lambda_1(c)$, $\lambda_2(c)$, $\lambda_1^*(c)$, $\lambda_2^*(c)$ split off the symmetry circle and form a quadruplet.

Howard and MacKay have described both cases in more detail (for $b=1$). In the case of mixed eigenvalues the formation of a quadruplet is typical. In almost any family of semi-symplectic matrices $M(c)$ a quadruplet is created (or destroyed) at a multiple mixed pair of eigenvalues. They also have shown that eigenvalues with equal signature typically do not collide at all.

All these results have originally been proven for the symplectic case $b=1$. We now give the argument why they must hold in the general semi-symplectic case.

For any semi-symplectic family $M(c)$ one can define a corresponding symplectic family $M_S(c)\equiv M(c)/\sqrt{b}$. Also the other way around: for any symplectic family $M_S(c)$ and for any given $0<b<1$ one can define a semi-symplectic family $M(c)\equiv M_S(c)\sqrt{b}$. Note that: (i) all eigenvalues of $M_S(c)$ are equal to those of $M(c)$, apart from a multiplication by $1/\sqrt{b}$; (ii) all eigenspaces and eigenprojections of M_S and M are identical; (iii) the signatures of the corresponding eigenvalues are identical.

Now if the semi-symplectic version of a symplectic lemma is not true, then we could construct a symplectic family that would violate the symplectic result. This would give a contradiction and thus we conclude that the semi-symplectic versions of the results of Krein (1955), Moser (1958) and Howard and MacKay (1987) must hold.

We conclude that pairs of complex eigenvalues on the symmetry circle of semi-symplectic matrices cannot leave this symmetry circle under small perturbations, as long as they are simple or multiple but definite. They almost certainly leave the symmetry circle and form a quadruplet in the case of multiple mixed eigenvalues.

5. Period-doubling sequence and Krein signatures in the uncoupled case

In order to understand the influence of weak coupling on the period-doubling sequence of a map F of the form (3.1), we first have to understand the uncoupled case properly. Especially we have to know the Krein signature of the eigenvalues of DF^Q at every stage of the period-doubling sequence ($Q=1,2,4,8\ldots$) in the uncoupled case. Therefore we investigate the relationship between the Krein signature of the complex eigenvalues of a general semi-symplectic map M and its square M^2. The squaring rule for signatures turns out to be very simple.

With this knowledge we investigate the behaviour of the Krein signature in proper period-doubling sequence (see section 3) of a 2D map like H_c. We have a distinguish between the conservative and the dissipative case, the former being more simple. The result, however, turns out to be the same: the Krein signature alternates in a proper period-doubling sequence.

Then we investigate the Krein signature of the eigenvalues of L_d^Q ($Q=1,2,4\ldots$). It turns out that this signature forms an effectively random sequence in Q (for almost all parameter values d).

We already know that in the case of weak coupling instabilities in the period-doubling sequence can only be created by collision of eigenvalues with opposite signature. So we end this paragraph with an investigation where and how eigenvalues collide in the uncoupled case. We will see that the eigenvalues can meet anywhere on the symmetry circle with any relative signature.

5.1. Squaring rule for signatures

Let M be any $(2n\times 2n)$ semi-symplectic matrix, with a pair of simple complex eigenvalues λ and $\lambda^*=b/\lambda$ $(\neq\lambda)$. Then M^2 is also semi-symplectic and has eigenvalues λ^2 and λ^{*2}. Unless $\lambda=\pm i\sqrt{b}$ and unless $(-\lambda)$ is an eigenvalue too, λ^2 and λ^{*2} form again a simple complex pair. So the signatures of both pairs of eigenvalues are well defined. Moreover, there is a very simple relationship between these signatures.

Lemma 5.1. Let M be a semi-symplectic matrix with a pair of simple complex eigenvalues λ and $\lambda^*=b/\lambda$ $(\neq\lambda)$ and $\lambda\neq\pm i\sqrt{b}$ and let $(-\lambda)$ be not an eigenvalue. Then

$$\text{signature}(\lambda^2,\lambda^{*2}) \text{ of } M^2=\text{sign}(\lambda+\lambda^*)\ \text{signature}(\lambda,\lambda^*) \text{ of } M. \tag{5.1}$$

Proof. First observe that λ and λ^2 have the same generalized eigenspaces. The real invariant plane W_λ in the definition of Krein signature is 2D in this case because the pair of eigenvalues is simple. If e, e^* are the eigenvectors of λ and λ^* respectively, then any vector $x\in W_\lambda$ can be written as $x=\xi e+\xi^* e^*$, $\xi\in\mathbb{C}$. Using the anti-symmetry of J (section 2) we calculate (x, JMx) for $x\in W_\lambda$ (we use the convention $(e,f)\equiv\Sigma\, e_j^* f_j$ for the inproduct of complex vectors):

$$(x, JMx)=\xi\xi^*\lambda^*(e, Je)+\xi\xi^*\lambda(e^*, Je^*)=\xi\xi^*(\lambda^*-\lambda)(e, Je)$$

where (e, Je) is purely imaginary (as it should be). The same calculation for (x, JM^2x) yields

$$(x, JM^2x)=\xi\xi^*(\lambda^{*2}-\lambda^2)(e, Je).$$

So we conclude that

$$(x, JM^2x)=(\lambda^*+\lambda)(x, JMx).$$

We know that (x, JMx) and (x, JM^2x) must be definite because every simple complex pair must be definite (lemma 4.1). Using the definition 4.2 of signature, this proves the lemma directly.

The lemma above does not describe the cases $\lambda=\pm i\sqrt{b}$ and $\lambda=\pm\sqrt{b}$. In the case that M has an eigenvalue $\lambda=\pm i\sqrt{b}$, M^2 has a double semisimple eigenvalue $-b$. So its signature is not defined. In the case that M has a semisimple double eigenvalue $\lambda=\pm\sqrt{b}$, signature (λ) is not defined. For the case in which M has a non-semisimple double eigenvalue $\pm\sqrt{b}$ we find lemma 5.2.

Lemma 5.2. Let M be a semi-symplectic matrix with a non-semisimple double eigenvalue $\lambda=\pm\sqrt{b}$. Then

$$\text{signature}(\lambda^2,\lambda^{*2}) \text{ of } M^2=\text{sign}(\lambda)\ \text{signature}(\lambda,\lambda^*) \text{ of } M. \tag{5.2}$$

Proof. First note that the Krein signature of λ is well defined (lemma 4.3) because λ is not semisimple. The same holds for λ^2. Now consider a smooth family of semi-symplectic matrices $M(\varepsilon)$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ such that $M(0) = M$, $\lambda(0) = \lambda^*(0) = \lambda = \pm\sqrt{b}$ and $\lambda(\varepsilon)$, $\lambda^*(\varepsilon)$ form a simple complex pair for $\varepsilon > 0$. Then the signature of $\lambda(\varepsilon)$, $\lambda^*(\varepsilon)$ is also well defined and the continuity result (lemma 4.3) assures that

$$\text{signature}(\lambda(0)) = \text{signature}(\lambda(\varepsilon), \lambda^*(\varepsilon)) \qquad \text{for } 0 < \varepsilon \leqslant \varepsilon_0.$$

The same holds for the squared eigenvalues. For $\varepsilon > 0$ we can apply lemma 5.1. Then taking $\lim \varepsilon \downarrow 0$ and again using the continuity we find lemma 5.2.

5.2. *Krein signature in a conservative 2D period-doubling sequence*

Consider the 2D symplectic mapping H_c that possesses a proper period-doubling sequence in the parameter c. Since the periodic orbits in a period-doubling sequence are spectrally stable, the eigenvalues must be on the unit circle during the whole period-doubling sequence. Since this sequence is proper, multiple eigenvalues ± 1 are not semisimple (section 3). So the signature of the two eigenvalues is defined for all values of the c-parameter.

Denote the parameter values of the period-doubling bifurcation by $c_0, c_1, c_2, \ldots$. Assume for convenience that $c_0 < c_1 < c_2 < \ldots$. When continuously changing $c: c_{q-1} \to c_q$ the eigenvalues move from $+1$ to -1 on the unit circle in a continuous way. Their Krein signature does not change in this operation because of the continuity of signatures (lemma 4.2). Thus the signature of the eigenvalues in a period-doubling sequence depends only on the period (thus not on c explicitly). We will show that they alternate in the period-doubling sequence.

The two eigenvalues collide at -1 as $c \to c_q$. At the bifurcation point $c = c_q$

$$(S_q(c_q))^2 = S_{q+1}(c_q) \tag{5.3}$$

where $S_q(c) = DH_c^{2^q}$ (fixed point of period 2^q). Now by lemma 5.2 we see that at a period-doubling bifurcation the signature of the period-doubled orbit must be opposite to the original signature:

$$\text{signature}(S_{q+1}(c_q)) = -\text{signature}(S_q(c_q)). \tag{5.4}$$

Again using the continuity of signature (lemma 4.2) we see that signature$(S_{q+1}(c)) =$ signature$(S_{q+1}(c_q))$ for $c > c_q$. Consequently the Krein signature alternates in a 2D conservative period-doubling sequence.

5.3. *Dissipative 2D case*

Consider a 2D dissipative, semi-symplectic map $H_{c,b}$ with $\det(DH) = b < 1$. Let $H_{c,b}$ have a proper period-doubling sequence in c. This dissipative case differs significantly from the conservative case because the eigenvalues of the stable periodic points can (and do) become real (figure 4). So the Krein signature is no longer defined during the whole period-doubling sequence (but only in parts of it).

However, in a proper period-doubling sequence there must be in each interval $I_q \equiv (c_{q-1}, c_q)$ precisely one interval $J_q \subset I_q$ such that the eigenvalues of the 2^q orbit are complex iff $c \in J_q$. There can only be one such an interval J_q since in a proper period-doubling sequence the eigenvalues depend monotonically on the bifurcation parameter (section 3). Krein signatures can be defined for all $c \in J_q$, and it is clear (by

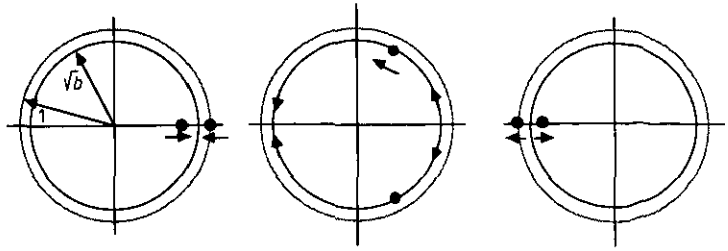

Figure 4. Behaviour of eigenvalues in one part of a 2D proper period-doubling sequence, $b<1$, from the saddle-node bifurcation (left) to the period-doubling bifurcation (right).

continuity of the signatures) that the signature does not change in J_q. We now argue that in many families $H_{c,b}$ the signature must alternate in the same way as in the conservative case.

We consider families of mappings $H_{c,b}$ that are semi-symplectic for all c and all $b \in [b_0, 1]$, $0<b_0<1$. So, varying c at every period-doubling bifurcation, precisely one period-doubled orbit is born. Thus for every c there exists precisely one unique stable periodic orbit that is a continuation of the original one. We call the collection of parameter values (c, b) at which the $2^q \to 2^{q+1}$ period-doubling bifurcation occurs L_q. Using the properties of proper period-doubling sequences we know that L_q is the graph of a continuous function over b: $L_q = \{(c, b) \mid c = c_q(b)\}$.

Varying b, we can also continue the stable periodic orbits uniquely. As long as there are no bifurcations (we do not cross any L_q) this follows from the implicit function theorem. But also varying b through a bifurcation value, the stable periodic orbit can be continued uniquely. To see this, change b from just before a bifurcation value to just after and consider an alternative route through parameter space as in figure 5 (first change c through L_q, then change b and then change c backwards). This alternative route must exist and can be chosen close to the original one because L_q is a graph over b. Along this alternative route the stable periodic orbit can clearly be continued uniquely into a stable period-doubled one. This uniqueness guarantees the unique continuation along the original route.

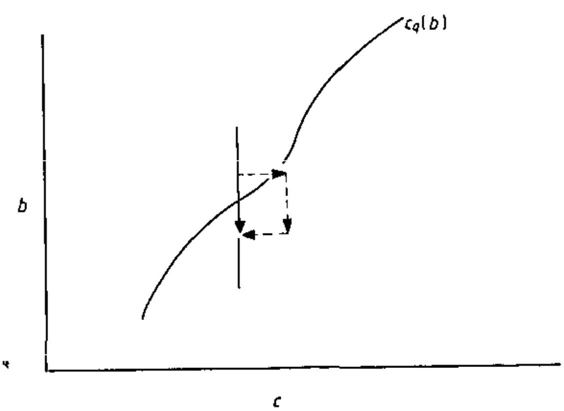

Figure 5. Part of the (c, b) parameter plane of a semi-symplectic family $H_{c,b}$. Period-doubling bifurcation occurs at the line $c=c_q(b)$. Full arrow, bifurcation by changing b only; broken arrow, alternative route that can be chosen arbitrarily close to the first one.

We conclude that $H_{c,b}$ possesses a continuous family of (spectrally) stable periodic orbits in (c, b) and that with each (c, b)-value there corresponds precisely one element of this family. With this knowledge we can describe the Krein signatures in a proper period-doubling sequence of a dissipative map.

Lemma 5.3. Let $H_{c,b}$ be a 2D semi-sympletic map for all c and all $b \in [b_0, 1]$ where $b = \det(DH)$ and $0 < b_0 < 1$. Suppose that $H_{c,b}$ possesses a proper period-doubling sequence in c for all $b \in [b_0, 1]$. Then:
(i) the Krein signature of the 2^q periodic orbit is for all $c \in J_q(b)$ the same and thus the signature depends only on q;
(ii) signature$(q+1) = -$signature(q).

Proof (see also figure 6). Let $V_q = \{(c, b) \mid b_0 \leqslant b \leqslant 1, c \in J_q(b)\}$. So if $(c, b) = (c_1, b_1) \in V_q$ then there exists a stable 2^q-periodic orbit with complex eigenvalues. So their Krein signature is then defined. By changing c we get $(c, b) = (c_2, b_1) \in V_{q+1}$ and we want to compare the signatures of the eigenvalues of the original orbit and the period-doubled orbit.

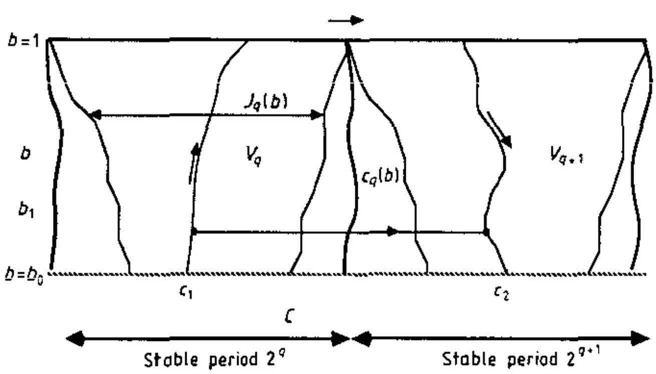

Figure 6. Part of the (c, b) parameter plane for a family $H_{c,b}$ of semi-symplectic mappings with a proper period-doubling sequence in c. Dotted areas V_q refer to the Jacobian with eigenvalues in a complex pair. Thick line indicates period-doubling bifurcations at $c = c_q(b)$. Arrows indicate the two routes used in the proof of lemma 5.3.

We do this by considering an alternative route from (c_1, b_1) to (c_2, b_1). First vary $b: b_1 \to 1$ and c such that $(c, b) \in V_q$ all the time. Then keep $b = 1$ and change c through the bifurcation value $c = c_q(1)$ so that $(c, b) \in V_{q+1}$. Then change $b: 1 \to b_1$ back and again c so that $(b, c) \in V_{q+1}$ all the time. We already have shown that we can continue the stable periodic orbits uniquely. During this whole procedure the eigenvalues of the stable orbit (first period 2^q, then period 2^{q+1}) are complex, so their Krein signature is defined during the whole route. This is also true at the bifurcation value $b = 1$, $c = c_q(1)$, for proper period-doubling sequences can have only non-semisimple double eigenvalues. Moreover, Krein signatures are conserved by continuous perturbations so that the signature can only change at the bifurcation value $b = 1$, $c = c_q(1)$. And we know from the conservative case that it necessarily must change sign there.

So we have created an alternative route from (c_1, b_1) to (c_2, b_1) and we know that for $(c, b) = (c_2, b_1)$ there must exist a period 2^{q+1} orbit, continuously deformed out of the original 2^q one, with signature opposite to that original one at (c_1, b_1). By the unicity of a continuous family of stable periodic orbits, the orbit we get by the alternative route must be the same as the one we get by directly changing $c: c_1 \to c_2$ and keeping $b = b_1$ fixed. So also by the direct route the signature in J_{q+1} must be opposite to that one in J_q.

5.4. Signature of iterates of a 2D linear map

In order to understand period-doubling sequences of F (equation (3.1)), we must also investigate the Krein signatures of the semi-symplectic $(L_d)^{2^q}$. We have assumed the eigenvalues λ of L_d to be complex, so that the eigenvalues λ_{2^q} of $(L_d)^{2^q}$ are like

$$\lambda_{2^q} = \sqrt{b^{(2^q)}}\exp(\mathrm{i}\pi\phi_q) \qquad 0 \leqslant \phi_q \leqslant 1 \tag{5.5}$$

and complex conjugate. From now on we will call ϕ_q the phase of λ_{2^q}. We investigate how the signatures of $\lambda_2, \lambda_4, \ldots, \lambda_{2^q}, \ldots$ depend on λ_1 and its signature.

The relation between the phases ϕ_{q-1} and ϕ_q of two eigenvalues $\lambda_{2^{q-1}}$ and λ_{2^q} is given by

$$\phi_q = T(\phi_{q-1})$$
$$T(\phi) = \begin{cases} 2\phi & \text{if } 0 \leqslant \phi \leqslant \frac{1}{2} \\ 2(1-\phi) & \text{if } \frac{1}{2} < \phi \leqslant 1. \end{cases} \tag{5.6}$$

The tentmap (5.6) does not only determine λ_{2^q}, given λ_1, but also its signature. To see this, we consider the well-known symbolic dynamics for T (Devaney 1986) as follows. With every $\phi \in [0, 1]$ we associate an infinite string $(a_0, a_1, a_2, \ldots)$ where the $a_n \in \{0, 1\}$ are defined by

$$a_n = \begin{cases} 0 & \text{if } 0 \leqslant T^n(\phi) < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < T^n(\phi) \leqslant 1 \\ \text{undefined} & \text{if } T^n(\phi) = \frac{1}{2}. \end{cases} \tag{5.7}$$

Apart from the ambiguities in points $\phi = p.2^{-q}$ $(p, q \in \mathbb{N})$ the connection between ϕ and its itinerary $(a_0, a_1, a_2, \ldots)$ is one-to-one (Devaney 1986). Consequently, with every infinite row $(a_0, a_1, a_2, \ldots)$ there corresponds a unique phase ϕ.

Now let $\lambda_1 = \sqrt{b}\exp(\mathrm{i}\pi\phi_0)$ and let ϕ_0 have itinerary $(a_0, a_1, a_2, \ldots)$. By (5.5), (5.6) and (5.7) the real part of λ_{2^q} is positive (negative) iff $a_q = 0$ (1). Moreover, this real part is zero iff a_q is undefined. Applying the squaring rule for signatures (lemma 5.1) we see, provided that a_{q-1} is defined,

$$\text{signature}(\lambda_{2^q}) = (-1)^{a_{q-1}}\,\text{signature}(\lambda_{2^{q-1}}). \tag{5.8}$$

Repeating this argument gives

$$\text{signature}(\lambda_{2^q}) = (-1)^{\sum_{n=0}^{q-1} a_n}\,\text{signature}(\lambda_1). \tag{5.9}$$

Almost all ϕ_0 have an infinite well-defined itinerary. Therefore, with (5.9) the signatures of the corresponding row of eigenvalues $(\lambda_1, \lambda_2, \lambda_4 \ldots)$ are well defined and determined by the itinerary of ϕ_0 and the signature of λ_1.

Only for phases $\phi_0 = p2^{-q}$ $(p, q \in \mathbb{N})$ is the infinite itinerary not defined. This set is of zero measure. If we take p uneven (always possible), only the first $a_0, \ldots, a_{q-2}$ are defined. Therefore the signatures of only $\lambda_1, \lambda_2, \lambda_4, \ldots, \lambda_{2^{q-1}}$ are defined. For $n \geqslant q$, $L^{(2^n)}$ is semisimple with a double eigenvalue on the negative $(n = q)$ or positive $(n \geqslant q+1)$ real axis.

The dynamics of the tentmap (5.6) are relatively simple (see Devaney 1986). For almost all starting points ϕ_0 it holds: (i) the orbit of ϕ_0 fills the whole interval (0, 1) uniformly; (ii) the values a_n in the itinerary of ϕ_0 are uncorrelated.

The exceptional starting points ϕ_0 are those with an asymptotically periodic itinerary (set of zero measure). Moreover, the dynamics of the tentmap depend sensitively on the starting point ϕ_0. The tentmap is even chaotic.

We summarize the results in the following lemma.

Lemma 5.4. Let L_d be a 2×2 semi-symplectic matrix with complex eigenvalues λ_1 and λ_1^*. Let λ_{2^q} and $\lambda_{2^q}^*$ be the eigenvalues of $L_d^{(2^q)}$ with phases $\phi_q\in[0,1]$ (see equation (5.5)). Then for almost all parameter values d: (i) the phases $(\phi_0, \phi_1, \ldots)$ are uniformly distributed over the interval $(0, 1)$; (ii) the signatures of $\lambda_1, \lambda_2, \ldots, \lambda_{2^q}, \ldots$ are well defined; (iii) the row of signatures is uncorrelated.

Summarizing the results on 2D mappings we see: for a 2D constant Jacobian map with a proper period-doubling sequence in parameter c, the Krein signature can be defined as long as the eigenvalues are complex; the signature depends only on c through the period and alternates in the period-doubling sequence.

The Krein signatures of the complex eigenvalues of the iterates of a 2D linear map form typically an uncorrelated sequence.

5.5. *Types of collisions of eigenvalues*

We now investigate where and how collisions of eigenvalues can occur in a period-doubling sequence of a 4D map (3.1) with no coupling ($k=0$). We will show that, typically, collisions occur anywhere on the symmetry circle and with any relative signature (equal or opposite).

In the uncoupled case, the position of collision of eigenvalues of a 2^q periodic orbit is completely determined by the (fixed) position of the eigenvalues of L^{2^q}. Because we have assumed the period-doubling sequence of H_c to be proper, the two pairs of eigenvalues must meet precisely once in each step of the period-doubling sequence.

Typically, the two pairs of eigenvalues collide somewhere on the symmetry circle off the real axis and create a complex pair of multiple eigenvalues. This complex pair can be definite or mixed, depending on the relative signature.

Exceptionally, the eigenvalues of $L_d^{2^q}$ can become real, i.e. $\pm(\sqrt{b})^{2^q}$. This only occurs for very special values of the parameter d, namely if the eigenvalues of L_d have phase ϕ of the form $p2^{-n}$ ($p, n\in\mathbb{N}$). In that case, a four-fold eigenvalue occurs in the period-doubling sequence.

Now consider the whole period-doubling sequence of F (equation (3.1)) for $k=0$ and given value for d. We assume d to be typical in the sense that the phase ϕ_0 of the eigenvalues of L_d does not have an asymptotically periodic itinerary. Then we know by the results above (lemma 5.4) that considering all periods together the position of collision of eigenvalues is uniformly distributed over the phases ϕ. Especially, by choosing the period 2^q properly, we can find a multiple eigenvalue arbitrarily close to the real axis (ϕ arbitrarily close to 0 or 1).

For these typical d-values, the Krein signature of one pair of eigenvalues alternates in the period-doubling sequence (lemma 5.3), while the signature of the other pair forms an uncorrelated row (lemma 5.4). So in the period-doubling sequence both equal and opposite signatures must occur. Especially, by choosing the period 2^q properly, we can find a multiple mixed eigenvalue arbitrarily close to the real axis. The same holds for a multiple definite pair.

We conclude that in the uncoupled case for almost all values of the parameter d Krein collisions of eigenvalues with opposite signature are inevitable and that these collisions occur anywhere on the symmetry circle, also arbitrarily close to the real axis. The same conclusion holds for equal signature.

6. Period-doubling sequence in the case of weak coupling

Now that we have described the period-doubling sequence in the uncoupled case, we can turn to the case with weak coupling. First note that turning on the coupling slightly can influence the position of the eigenvalues only slightly. So collisions of eigenvalues in the case with weak coupling occur close to those of the uncoupled case, as listed above. The most common cases are collisions of two complex pairs with equal or opposite signatures. Howard and MacKay (1987) have already examined what can happen in these two cases.

When two equal-signed eigenvalues move towards each other, they typically do not collide, but repel each other when close together. We see an 'avoided crossing' (figure 7). By Krein's theorem we know that the eigenvalues can never leave the symmetry circle, even if they do collide. So collisions into definite eigenvalues cannot affect the stability of the periodic orbit.

When two opposite-signed eigenvalues move towards each other, they typically do collide and form a quadruplet (figure 8). So collisions into mixed eigenvalues can affect the stability of the periodic orbit.

We will calculate the unfoldings of a multiple complex eigenvalue explicitly in two examples, one with opposite signatures and one with equal signatures. We will show

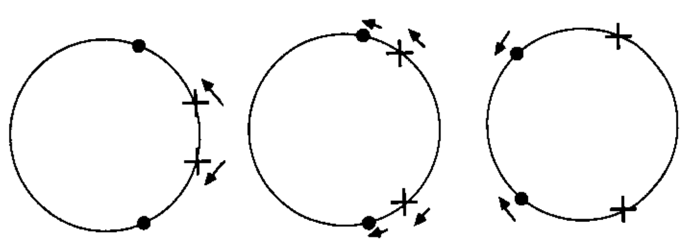

Figure 7. Typical behaviour of eigenvalues in one part of the period-doubling sequence of a symplectic map ($b = 1$) with non-zero coupling in the case of two complex pairs of eigenvalues with equal signature. The eigenvalues do not collide at all: an 'avoided crossing'.

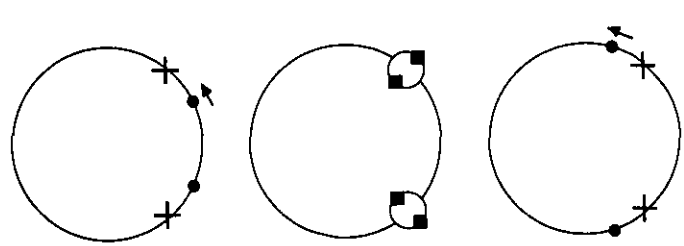

Figure 8. Typical behaviour of eigenvalues in one part of the period-doubling sequence of a symplectic map ($b = 1$) with non-zero coupling in the case of two complex pairs of eigenvalues with opposite signature. The eigenvalues form temporarily a quadruplet: a 'bubble of instability'.

that the results of these calculations are typical in the class of studied maps (3.1). We will see that also higher iterates of maps of type (3.1) are typically described by our two examples, at least in lowest order approximation. This gives us the opportunity to study period-doubling sequences of maps of type (3.1) in the next section.

6.1. Unfolding of mixed eigenvalues in an example

As our first example, we choose a map F of type (3.1) with

$$H_c(y) = \begin{bmatrix} 2cy_1 + 2y_1^2 - by_2 \\ y_1 \end{bmatrix} \qquad L_d = \begin{bmatrix} 2d & b \\ -1 & 0 \end{bmatrix}$$
$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.1}$$

The origin is a fixed point with opposite-signed eigenvalues. Let $c = c_0$ and $d = d_0$ be such that $DF(0)$ possesses a pair of double complex eigenvalues λ_0 and λ_0^* on the symmetry circle if $k = 0$ $(-1 \leqslant c_0 = d_0 \leqslant +1)$.

In appendix 2 we calculate the influence of perturbations on c, d and k on these eigenvalues, by first calculating the influence on the stability indices ρ (equation (2.5)). We find that a perturbation $c = c_0 + \delta c$, $d = d_0 + \delta d$, $k \neq 0$ gives

$$\rho = \rho_0 + \delta c + \delta d \pm \sqrt{(\delta c - \delta d)^2 - k^2} \tag{6.2}$$

where ρ_0 is the stability index belonging to $c = c_0$, $d = d_0$, $k = 0$. We know (cf equation (2.5)) that ρ is complex iff the corresponding eigenvalues form a quadruplet. So a quadruplet exists iff $(\delta c - \delta d)^2 < k^2$.

For the eigenvalues λ, equation (6.2) implies in lowest-order perturbation (appendix 2)

$$\lambda = \lambda_0 + \frac{\lambda_0}{\lambda_0 - b/\lambda_0}\left(\delta c + \delta d \pm \sqrt{(\delta c - \delta d)^2 - k^2}\right). \tag{6.3}$$

Varying (decreasing) the bifurcation parameter c and keeping d and $k \neq 0$ fixed $(\delta d = 0)$, we see that a quadruplet is created out of two pairs when $\delta c = |k|$ and that this quadruplet falls apart in two pairs again when $\delta c = -|k|$. We observe a 'bubble of instability' (figures 8 and 9) (Howard and MacKay 1987) obeying in lowest order

$$|\lambda - \lambda_0|^2 = \frac{bk^2}{[2\,\mathrm{Im}(\lambda_0)]^2}. \tag{6.4}$$

Note that the size of this bubble is larger when λ_0 is closer to the real axis.

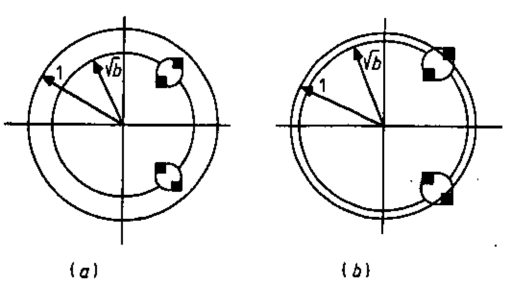

Figure 9. 'Bubble of instability' for a semi-symplectic map ($b < 1$). (a) Relatively small coupling (stability conserved). (b) Relatively large coupling: loss of stability.

Only for small enough coupling, all the pairs of eigenvalues involved with a bubble of instability are complex pairs (as in figures 8 and 9). This is the case whenever the inequality

$$|k| < 2\sqrt{b} \pm \rho_0 \tag{6.5}$$

holds for both $\pm$ signs. To see this, we calculate ρ at the points of creation/annihilation of the quadruplet (i.e. $\delta d = 0$, $\delta c = \pm|k|$) with equation (6.2). By equation (2.5) we know that the created/annihilated pairs are complex iff $|\rho| < 2\sqrt{b}$. This gives inequality (6.5) directly.

If the coupling strength is larger, so that inequality (6.5) is violated for at least one sign, the bubble of instability has become so large as to hit the real axis (figures 10 and 11). The smallest value for $|k|$ for which this happens, $k_v(\lambda_0)$, is lower when λ_0 is closer to the real axis (for then $2\sqrt{b} \pm \rho_0$ is small for one sign). In the limit for $\mathrm{Im}(\lambda_0)$ to zero, $k_v(\lambda_0)$ also goes to zero.

From these calculations we see the following.

In the conservative case the fixed point becomes unstable for any non-zero coupling strength. Normally (i.e. for collision far from ± 1) the behaviour of eigenvalues, when varying c (and fixing k and d), is as in figure 8: for $|\delta c| < k$ there is a bubble of instability, but varying δc through k the eigenvalues reconcile and the fixed point

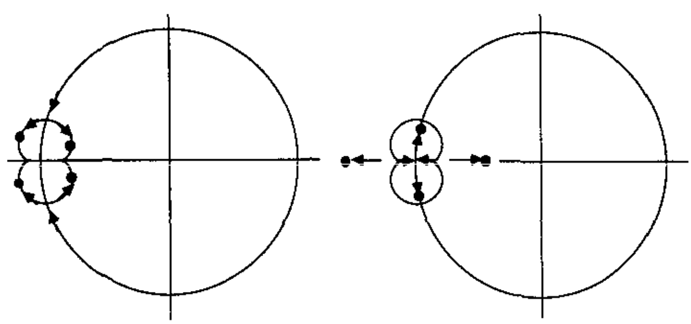

Figure 10. Collision of two pairs of complex eigenvalues of a symplectic map ($b = 1$) with opposite signatures in the neighbourhood of -1. Note that two eigenvalues pass -1 from the real axis to the unit circle, which is essentially different from figure 8.

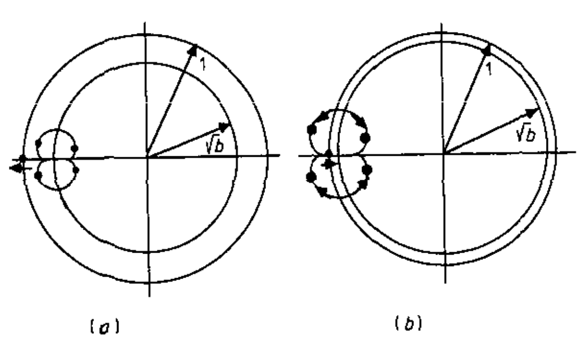

Figure 11. Collision of two pairs of eigenvalues of a semi-symplectic map ($b < 1$) with opposite signature and in the neighbourhood of -1. (a) Relatively small coupling: stability is conserved; period-doubling bifurcation is conventional. (b) Relatively large coupling: loss of stability; eigenvalue moves inwards the unit circle at the bifurcation (as in figure 10).

regains its stability. However, if the collision of eigenvalues take place near -1, the eigenvalues hit the real axis before reconciling (figure 10). A more complex bifurcation then happens. We see that once the origin has lost its stability, it will never regain it.

The dissipative case is slightly different because the unit circle and the symmetry circle no longer coincide. This means that a bubble does not cause instability at once: only for strong enough coupling (k above some threshold) the fixed point loses its stability (figure 9). Again the eigenvalues reconcile, unless they hit the real axis before. If they hit the real axis, they can do so between $-\sqrt{b}$ and -1 (figure 11) or outside -1 (figure 11). This last case is the only case in which the origin loses its stability but never regains it.

The same conclusions hold *mutatis mutandis* for eigenvalues near to $+1$ and $+\sqrt{b}$.

6.2. *Unfolding of definite eigenvalues in an example*

The calculations for the definite eigenvalues are very much like those for the mixed. The results differ only in some signs, which, however, generates essentially different behaviour.

As our second example, we take H_c and M as in equation (6.1), but choose

$$L_d = \begin{bmatrix} 2d & -b \\ 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.6}$$

The origin is a fixed point with equal-signed eigenvalues. Let $c = c_0$ and $d = d_0$ be such that $DF(0)$ possesses a pair of double complex eigenvalues λ_0 and λ_0^* on the symmetry circle if $k=0$ (thus $-1 \leqslant c_0 = d_0 \leqslant +1$).

In the same way as in the mixed case, we find for small perturbations

$$\rho = \rho_0 + \delta c + \delta d \pm \sqrt{(\delta c - \delta d)^2 + k^2} \tag{6.7}$$

where ρ_0 is the stability index belonging to $c = c_0$, $d = d_0$, $k=0$. Note that since the expression under the root is non-negative, a quadruplet can never be created (cf equation (2.5)).

For the eigenvalues λ, equation (6.7) implies in lowest-order perturbation

$$\lambda = \lambda_0 + \frac{\lambda_0}{\lambda_0 - b/\lambda_0}\left(\delta c + \delta d \pm \sqrt{(\delta c - \delta d)^2 + k^2}\right). \tag{6.8}$$

Varying the bifurcation parameter c and keeping d and k fixed, we see an 'avoided crossing', that is, the eigenvalues only approach each other, but do not collide. One may say that they 'push each other away'. When the distance between the eigenvalues is minimal, it holds

$$|\lambda - \lambda_0|^2 = \frac{bk^2}{[2\,\mathrm{Im}(\lambda_0)]^2}. \tag{6.9}$$

No quadruplets can be created and thus stability cannot be lost by formation of a quadruplet. The only way the fixed point can lose its stability is by formation of a real pair out of a complex pair at $\pm\sqrt{b}$ so that one eigenvalue can pass ± 1.

6.3. *Unfoldings of eigenvalues of fixed points in the general case*

The results from the example above turn out to be typical for unfoldings of multiple eigenvalues of a fixed point at the origin in the class of maps (equation (3.1)). Only the semi-symplecticity and the block structure (3.1) and the type of eigenvalue (definite of mixed) are essential. Using these properties, one can prove (appendix 3, equation (A3.12)) that equations (6.2) and (6.7) for fixed points of maps (3.1) at the origin in general read exactly

$$\rho=\rho_0+\tfrac{1}{2}(\delta\sigma(DH)+\delta\sigma(L)\pm\sqrt{(\delta\sigma(DH)-\delta\sigma(L))^2+4k^2\sigma(MN)}) \tag{6.10}$$

where σ means trace, δ denotes perturbation and DH should be evaluated in the fixed point 0. It is clear that this is a generalization of equations (6.2) and (6.7). Moreover, the coupling strength $\sigma(MN)$ is typically non-zero. Its sign is determined by the type of multiple eigenvalue: a + sign corresponds with a definite eigenvalue (equation (6.7)) and a − sign with a mixed eigenvalue (equation (6.2)). We note that the unfolding described by equation (6.10) is codimension two, which is in agreement with the results of Howard and MacKay (1987).

We conclude that the unfolding of a pair of double complex eigenvalues on the symmetry circle of a fixed point at the origin of any map of type (3.1) is essentially described by equation (6.2) or (6.7), depending on the relative signatures.

6.4. Unfoldings for eigenvalues of periodic fixed points

In deriving equation (6.10) we only used the special form (3.1) of the map F. Because iterates of semi-symplectic maps are again semi-symplectic, one might expect that the same argument will do for iterates of F. However, for two reasons, iterates of maps F of type (3.1) are themselves not of this type: (i) the position of the periodic fixed points depends on the parameters c, d and, more importantly, on k; (ii) the Jacobian matrix is only approximately of the desired block structure (3.1).

Nevertheless we will see that equation (6.2) or (6.7) is still essentially correct.

First we note that any q-periodic point $x_q(c, d, k)$ of a map (3.1) must be of the form

$$x_q(c,d,k)=\begin{bmatrix} y_q(c,d,k^2) \\ k\zeta_q(c,d,k^2)\end{bmatrix}. \tag{6.11}$$

This follows by introducing a new variable ζ as $z=k\zeta$. With the form (3.1) the mapping becomes

$$y'=H_c(y)+k^2M\zeta \qquad \zeta'=L_d\zeta+Ny. \tag{6.12}$$

In these coordinates the mapping depends only on the parameters c, d and k^2. This implies equation (6.11).

The position dependence of the Jacobian DF enters only via the y-component and thus it must be even in k. Therefore, the Jacobian evaluated in a periodic point $x_q=x_q(c, d, k)$ must be of the form

$$DF(x_q)=\begin{bmatrix} DH(c,d,k)^2 & kM \\ kN & L\end{bmatrix}. \tag{6.13}$$

For the relevant Jacobian of the q-fold iterated map this implies

$$\begin{aligned} DF^q(x_q)&=\begin{bmatrix} A_1(c,d,k^2) & kA_2(c,d,k^2) \\ kA_3(c,d,k^2) & A_4(c,d,k^2)\end{bmatrix} \\ &=\begin{bmatrix} A_1(c,d,0) & 0 \\ 0 & A_4(c,d,0)\end{bmatrix}+k\begin{bmatrix} 0 & A_2(c,d,0) \\ A_3(c,d,0) & 0\end{bmatrix}+\mathrm{O}(k^2) \end{aligned} \tag{6.14}$$

where A_j are 2×2 matrices (still depending on the period q).

We are now able to consider a q-periodic point with multiple eigenvalues. Let $c=c_0$ and $d=d_0$ be such that DF^q possesses multiple eigenvalues on the symmetry circle for $k=0$. We can calculate the influence of a perturbation $c=c_0+\delta c$, $d=d_0+\delta d$, $k\neq 0$ in appendix 3. We find that in lowest order (equation (A3.11))

$$\delta\rho=\frac{s_1\,\delta c+s_4\,\delta d\pm\sqrt{(s_1\,\delta c-s_4\,\delta d)^2+k^2U_0}}{2} \tag{6.15}$$

where s_1 and s_4 are non-zero constants and the constant U_0 is typically non-zero. Its sign is related to the type of eigenvalues: a plus sign describes the unfolding of a definite pair of eigenvalues and a minus sign that of a mixed pair.

This formula is essentially (up to a reparametrization) the same as in the period 1 case (6.2) and (6.7). Note, however, that those results are exact, whereas equation (6.15) is an approximation. So for small perturbations δc, $\delta d=0$, k all the results for the period 1 case (cf equations (6.4), (6.5) and (6.9)) also hold for the q-periodic orbits. Of course one should replace the friction and coupling parameters b and k by the effective friction and coupling:

$$b_{\text{eff}}(q)=b^q \qquad k_{\text{eff}}(q)=\sqrt{|U_0(q)|}k. \tag{6.16}$$

It is not clear how the effective coupling depends on the period. Especially the behaviour for $q\to\infty$ seems to be relevant: does the effective coupling strength tend to zero ($|U_0(q)|\to 0$ for $q\to\infty$) or is it uniformly bounded away ($|U_0(q)|>\text{constant}>0$ for q large)?

We have investigated this numerically for our example (equation (6.6)) up to period $q=64$. For each period $q=1$, 2, 4, 8, 16, 32, 64 we choose the parameters c and d such that for $k=0$ the periodic orbit had a purely imaginary multiple definite pair of eigenvalues. Turning on the coupling ($\delta c=\delta d=0$, $k\neq 0$) (6.15) gives for the two stability indices

$$|\rho_1-\rho_2|^2=|U_0|k^2+\text{higher-order terms}. \tag{6.17}$$

We have used this to determine $|U_0(q)|$ for $b=0.9$ and $b=0.99$. We found that $|U_0(q)|$ obeys by good approximation a power law behaviour:

$$|U_0(q)|\approx 4q^\alpha \qquad (q=1,2,4,8,\ldots,64) \tag{6.18}$$

with exponent α about 2.5. This strongly suggests that $|U_0(q)|$ is an increasing function of q and thus is uniformly bounded away from zero. The effective coupling seems to get stronger with the period.

7. Consequences for infinite period-doubling sequences

It is now clear what happens to the original infinite period-doubling sequence (of the uncoupled case), when the coupling is made non-zero. We have to distinguish between the conservative and the dissipative case. The conclusion will be that in the conservative case any finite coupling strength changes essential parts of the period-doubling sequence essentially, while in the dissipative case the same holds if the coupling is above some threshold value.

In the conservative case ($b=1$) with finite coupling strength k, at every period 2^q the eigenvalues interact just once. In half of the cases (on the average) the eigenvalues have equal signature, so the stability of the periodic orbit is not affected. In half of the cases (on the average) the eigenvalues have opposite signature, so that the periodic orbit loses its stability by a Hamiltonian Hopf bifurcation. In most cases (for small k) these Krein collisions take place just somewhere on the unit circle and a bubble of instability like in figure 8 occurs: changing the bifurcation parameter c the periodic orbit first loses its stability but later regains it. This process is followed by a normal period-doubling bifurcation at -1.

However, for almost all values of the d-parameter there are periods 2^q for which the Krein collision takes place very close to the real axis. More precisely, for any finite coupling strength k we can find a suitable period such that the bubble of instability circumvents the point -1, as in figure 10. The quadruplet reconciles in two real pairs. Therefore, the period-doubling bifurcation that follows creates an unstable period-doubled orbit (figure 12). Changing the bifurcation parameter c further, this unstable orbit regains its stability, by first creating a quadruplet that falls apart in two complex pairs (figure 12). This follows from equation (6.15) for the period-doubled orbit. The bifurcation diagram looks like figure 13. A crucial part of the period-doubling sequence, namely one of the period-doubling bifurcations, corresponds with an unstable orbit and thus is experimentally spoken of as invisible (figure 14). We cannot speak anymore of a period-doubling sequence in the usual way. We remark that the first period in which a Krein collision close to the real axis occurs depends rather sensitively on the parameters (that is, on d and k).

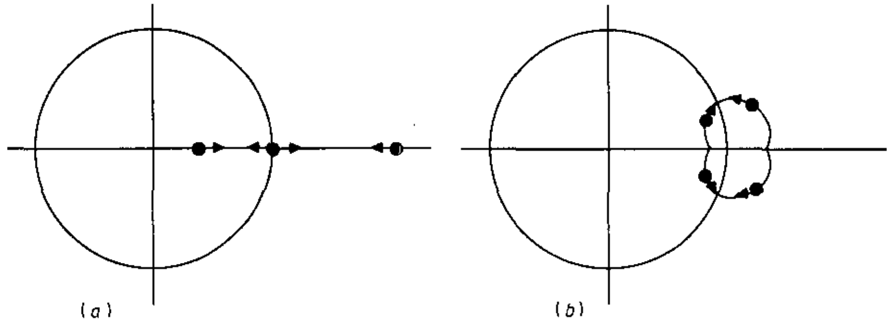

Figure 12. Position of the eigenvalues of the period-doubled orbit that arises from the process sketched in figure 10. (a) Moment of creation. Arrows indicate change of position of eigenvalues when the bifurcation parameter c is changed further. The eigenvalues will collide and form a quadruplet. (b) Changing the bifurcation parameter c further, the quadruplet falls apart in two complex pairs. At that moment the period-doubled orbit becomes stable (for the first time).

The dissipative case is much like the conservative one, only the unit circle and the semi-symplectic symmetry circle are no longer the same. A Krein collision of mixed eigenvalues will not necessarily lead to instability. Only for strong enough coupling do eigenvalues cross the unit circle (figures 9 and 15). We then have a Hopf bifurcation, creating an *invariant circle*.

Just as in the conservative case, the periodic orbit usually regains its stability when the parameter c is changed a little further. The quadruplet can reconcile in two complex pairs again, in two real pairs inside the unit circle or in two real pairs with two eigenvalues outside the unit circle. Only in this last case does the following period-doubling bifurcation create an (initially) unstable period-doubled orbit.

However, there is a practical difference that in dissipative systems hysteresis effects may occur. The invariant circle, created by the Hopf bifurcation, can be asymptotically

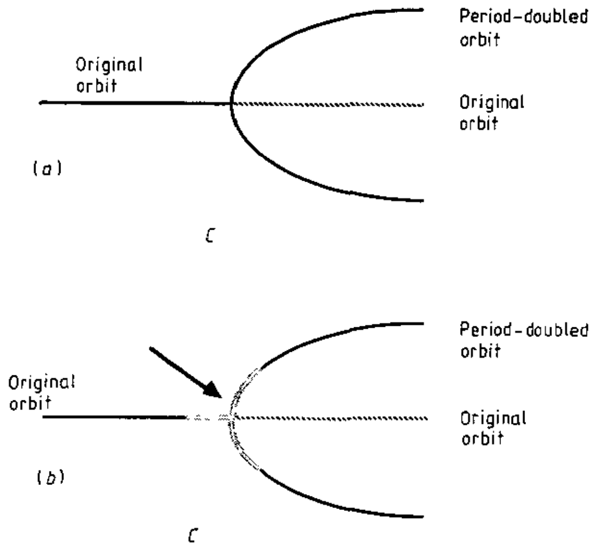

Figure 13. Bifurcation diagram for the period-doubling bifurcation of a symplectic map with all four eigenvalues close to -1. —— Stable period orbit; ········ unstable period orbit with one eigenvalue outside the unit circle; unstable periodic orbit with two eigenvalues outside the unit circle. (*a*) No coupling, $k=0$; ordinary period-doubling bifurcation. (*b*) With coupling so large that eigenvalues circumvent -1 (as in figure 10); period-doubled orbit is unstable just after its birth.

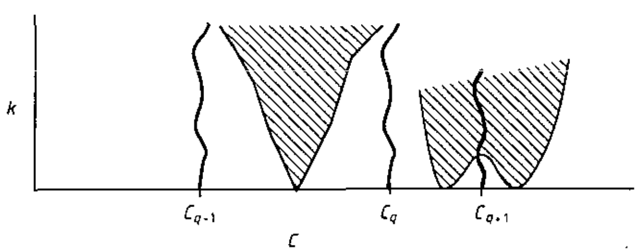

Figure 14. Sketch of a part of the (c, k) parameter plane for a symplectic map ($b=1$, d fixed). C_q indicates the bifurcation parameter value if $k=0$. Hatched areas refer to periodic orbits that are instable due to bubbles. Unhatched areas refer to periodic orbits that are stable. Thick lines refer to period-doubling bifurcations. Left, ordinary bubble of instability. Right, bubble that circumvents -1 for strong enough coupling. Note that a thick line (bifurcation) lies in the hatched area (instability).

stable. If this is the case and we vary the bifurcation parameter c adiabatically, we will stick to this invariant attracting circle. The moment the original periodic orbit regains its stability will thus not be noticed. So in the dissipative case, if the coupling strength is above some threshold value, we are also not able to follow the period-doubling sequence in practice (i.e. adiabatically).

Acknowledgment

I would like to thank T P Valkering for the many discussions that contributed to every part of this paper, and E W C van Groesen and Ruud van Damme for critically reading the manuscript.

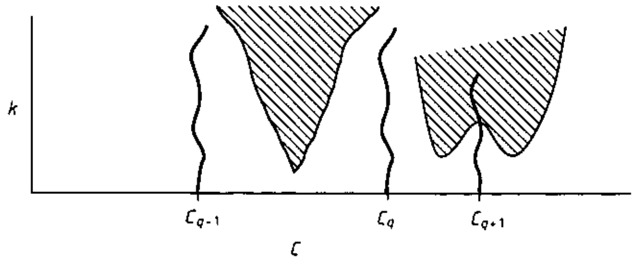

Figure 15. Sketch of a part of the (c, k) parameter plane for a semi-symplectic map ($b<1$, d fixed). C_q indicates the bifurcation parameter value if $k=0$. Hatched areas refer to periodic orbits that are instable due to bubbles. Unhatched areas refer to periodic orbits that are stable. Thick lines refer to period-doubling bifurcations. Left, ordinary bubble of instability. Right, bubble that circumvents -1 for strong enough coupling. Note that a thick line (bifurcation) lies in the hatched area (instability).

Appendix 1

In this appendix, the precise formulation of the first three lemmas of section 4 and their proofs are given.

Lemma 4.1. Let M be a semi-symplectic matrix (with respect to J, b) with a pair of (algebraically) simple complex eigenvalues λ, $\lambda^*=b/\lambda \neq \lambda$ (thus on the symmetry circle). Then the pair λ, λ^* is definite.

Proof. M is semi-symplectic, so $M_S \equiv M/\sqrt{b}$ is symplectic. If M has a pair of simple complex eigenvalues λ, λ^* with combined real invariant plane W_λ, then M_S has a pair of simple complex eigenvalues $\lambda_S=\lambda/\sqrt{b}$, $\lambda_S^*=1/\lambda_S$ with the same invariant plane W_λ. Now simple complex eigenvalues of symplectic matrices ($b=1$) are known to be definite (Yakubovich and Starzhinskii 1975), so (x, JM_Sx) is definite on $W_\lambda\backslash\{0\}$. But this implies directly that (x, JMx) is definite on $W_\lambda\backslash\{0\}$, so λ, λ^* must be definite too.

The Krein signature of complex pairs of simple eigenvalues has an important continuity property.

Lemma 4.2. The Krein signature of a pair of simple complex eigenvalues of a semi-symplectic matrix cannot change due to continuous perturbations, as long as this pair remains simple and complex.

Proof. As long as the pair of eigenvalues remain simple and complex, their signature is defined (lemma 4.1). So the quantity (x, JMx) of definition 4.1 must be definite on W_λ. This quantity clearly depends continuously on M (Kato 1966), so its sign cannot change.

Lemma 4.3. Let M be a semi-symplectic matrix with a double non-semisimple eigenvalue $\lambda=\pm\sqrt{b}$. Then λ is definite.

Moreover, let $M(\varepsilon)$ be a continuous family of semi-symplectic matrices with a pair of simple complex eigenvalues $\lambda(\varepsilon)$, $\lambda(\varepsilon)^*$ on the symmetry circle for $\varepsilon>0$ such that $M(0)=M$ and $\lambda(0)=\lambda$. Then λ has the same signature as $\lambda(\varepsilon)$ ($\varepsilon>0$).

Proof. In W_λ we choose a real basis e_1, e_2 consisting of the generalized eigenvectors of M, i.e.

$$Me_1 = \lambda e_1 \qquad Me_2 = \lambda e_2 + e_1.$$

Because M is not semisimple, $(y, J(M - \lambda \cdot \mathbb{1})x)$ cannot be zero for all $x, y \in W_\lambda$. When we write $x = x_1 e_1 + x_2 e_2$, and likewise for y we see that

$$(y, J(M - \lambda \cdot \mathbb{1})x) = x_2 y_2 (e_2, Je_1).$$

This can only be not identically zero if

$$(e_2, Je_1) \neq 0.$$

With this property we see that

$$(x, JMx) = (x_2)^2 (e_2, Je_1)$$

is semi-definite on W_λ and not identically zero. So by definition 4.3 λ is definite.

To prove the second part, first note that for $\varepsilon > 0$ the complex pair $\lambda(\varepsilon)$, $\lambda^*(\varepsilon)$ is definite by lemma 4.1. Thus $(x, JM(\varepsilon)x)$ is definite on $W_\lambda(\varepsilon)$ for $\varepsilon > 0$, and its sign is signature (λ, λ^*). Or in an alternative formulation, let $P(\varepsilon)$ be the eigenprojection on $W_\lambda(\varepsilon)$, then $(P(\varepsilon)x, JM(\varepsilon)P(\varepsilon)x)$ is semi-definite on whole $\mathbb{R}^{2n}$ (for $\varepsilon > 0$) and when it is non-zero, its sign is signature(λ, λ^*) (note that it cannot be identically zero!). The eigenprojection $P(\varepsilon)$ depends continuously on ε because $M(\varepsilon)$ depends continuously on ε (Kato 1966). Moreover, $P(0)$ is precisely the eigenprojection on $W_\lambda(0)$ (Kato 1966). Taking $\lim \varepsilon \downarrow 0$ we find that $(P(0)x, JM(0)P(0)x)$ is semi-definite on $\mathbb{R}^{2n}$ and thus that $(x, JM(0)x)$ is semi-definite on $W_\lambda(0)$. We already know by the first part of this lemma that $(x, JM(0)x)$ is not identically zero on W_λ, so its sign = signature λ is well defined and must be the same as signature$(\lambda(\varepsilon), \lambda^*(\varepsilon))$ for $\varepsilon > 0$.

Appendix 2

In this appendix we calculate the unfoldings of multiple mixed eigenvalues. We do this explicitly for our example, the map (6.1).

Consider the map F (6.1) and its fixed point in the origin. The eigenvalues of the linearization around this fixed point obey by semi-symplecticity

$$\begin{aligned} &\lambda^4 - \sigma\lambda^3 + \gamma\lambda^2 - b\sigma\lambda + b^2 = 0 \\ &\sigma = \mathrm{trace}(DF) \\ &\gamma = \text{second minor}(DF) = \{[\mathrm{trace}(DF)]^2 - \mathrm{trace}(DF^2)\}/2. \end{aligned} \tag{A2.1}$$

Introducing the stability indices $\rho = \lambda + b/\lambda$ (2.5), this reduces to

$$\rho^2 - \sigma\rho + \gamma - 2b = 0. \tag{A2.2}$$

We will investigate the influence of small perturbations in the parameters c, d and k, starting with a decoupled system $(k = 0)$ and with a complex pair of double eigenvalues (thus with $c_0 = d_0$, $|d_0| < \sqrt{b}$).

Uncoupled it holds (with equation (6.1))

$$\sigma_0 = 2(c_0 + d_0) \qquad \rho_0 = 4c_0 d_0 + 2b. \tag{A2.3}$$

Perturb $c = c_0 + \delta c$, $d = d_0 + \delta d$, $k \neq 0$; then

$$\sigma = \sigma_0 + 2(\delta c + \delta d) \qquad \gamma = \gamma_0 + 4(c_0 \delta d + d_0 \delta c) + 4\delta c \delta d + k^2. \tag{A2.4}$$

Substituting $c_0 = d_0$ we find for $\delta\rho = \rho - \rho_0$:

$$\delta\rho = \delta c + \delta d \pm \sqrt{(\delta c - \delta d)^2 - k^2} \tag{A2.5}$$

$\delta\rho$ (and thus ρ) complex implies a quadruplet (section 2). Thus there exists a quadruplet iff $k^2 > (\delta c - \delta d)^2$. Consider this case, then

$$\delta\rho = \delta c + \delta d \pm \mathrm{i}\sqrt{k^2 - (\delta c - \delta d)^2}. \tag{A2.6}$$

With this result on the stability index we can find a result for the eigenvalues λ. For by definition of ρ (equation (2.5))

$$\lambda^2 - \rho\lambda + b = 0. \tag{A2.7}$$

With $\lambda = \lambda_0 + \delta\lambda$, $\rho = \rho_0 + \delta\rho$ and $\rho_0 = \lambda_0 + b/\lambda_0$ this implies for $\delta\lambda$

$$(\delta\lambda)^2 + (\lambda_0 - b/\lambda_0 - \delta\rho)\delta\lambda - \lambda_0\delta\rho = 0. \tag{A2.8}$$

If

$$\left|\frac{\lambda_0 + b/\lambda_0}{(\lambda_0 - b/\lambda_0)^2}\,\delta\rho\right| \ll 1 \tag{A2.9}$$

and

$$|\delta\rho| \ll |\lambda_0 - b/\lambda_0|$$

we can approximate

$$\delta\lambda = \frac{\lambda_0}{\lambda_0 - b/\lambda_0}\,\delta\rho + \mathrm{O}(\delta\rho^2). \tag{A2.10}$$

Note that for λ_0 in a complex pair $b/\lambda_0 = \lambda_0^*$ and thus $\lambda_0 - b/\lambda_0 = 2\mathrm{i}\,\mathrm{Im}(\lambda_0) \neq 0$. All together, when varying c ($\delta c \neq 0$) and fixing d ($\delta d = 0$) we find a bubble of instability:

$$|\lambda - \lambda_0|^2 = \frac{k^2}{(2\,\mathrm{Im}\,\lambda_0)^2} + \mathrm{O}(\delta\rho^2). \tag{A2.11}$$

Appendix 3

In this appendix we perform some calculations on the eigenvalues of a general 4×4 semi-symplectic matrix A. Then we apply this to the Jacobi matrix of fixed and periodic points of maps F of type (3.1). We calculate the unfolding of a complex pair of double eigenvalues on the symmetry circle. It will be shown that the results for our example, as analysed in appendix 2, are typical for both fixed and periodic points of maps out of class (3.1).

First consider a general 4×4 semi-symplectic matrix A. We divide A in four 2×2 blocks:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \tag{A3.1}$$

Due to the semi-symplecticity, the eigenvalues of A are completely determined by $\det(A) = b$ and the first two minors of A, denoted $\sigma(A)$ and $\gamma(A)$, where $\sigma(A)$ is trace(A) and in general (for any matrix A)

$$\gamma(A) = (\sigma(A)^2 - \sigma(A^2))/2. \tag{A3.2}$$

Again we reduce the secular equation for the eigenvalues λ to an equation for the stability indices ρ (cf equation (A2.2)):

$$\rho^2 - \sigma(A)\rho + \gamma(A) - 2b = 0. \tag{A3.3}$$

We calculate $\sigma(A)$ and $\gamma(A)$. By the block division of A one finds

$$\sigma(A) = \sigma(A_1) + \sigma(A_4) \qquad \sigma(A^2) = \sigma(A_1^2) + \sigma(A_4^2) + 2\sigma(A_2A_3). \tag{A3.4}$$

Using equation (A3.2) the other way around

$$\sigma(A_1^2) = \sigma(A_1)^2 - 2\gamma(A_1) = \sigma(A_1)^2 - 2\det(A_1)$$

so that

$$\begin{aligned} \sigma(A) &= \sigma(A_1) + \sigma(A_4) \\ \gamma(A) &= \gamma(A_1) + \gamma(A_4) + \sigma(A_1)\sigma(A_4) - \sigma(A_2A_3) \\ &= \det(A_1) + \det(A_4) + \sigma(A_1)\sigma(A_4) - \sigma(A_2A_3). \end{aligned} \tag{A3.5}$$

So the value of the stability index ρ and of the eigenvalues λ of a 4×4 semi-symplectic matrix A are completely determined by $\sigma(A_1)$, $\sigma(A_4)$, $\sigma(A_2A_3)$, $\gamma(A_1)$, $\gamma(A_4)$ and b.

Now let $A(c, d, k) = DF^q(x_q(c, d, k))$ where F is a map of type (3.1) and $x_q = x_q(c, d, k)$ is a q-periodic point for F. We investigate the case that DF^q has a complex pair of double eigenvalues on the symmetry circle and small perturbations of this case. Let $A = DF^q$ have such a complex pair of double eigenvalues for parameters $c = c_0$, $d = d_0$, $k = 0$. For these parameter values the off-diagonal blocks A_2 and A_3 are identically zero. So there can be a pair of double eigenvalues only if $\sigma_0(A_1) = \sigma_0(A_4)$ where the subscript 0 refers to the special parameter choice $c = c_0$, $d = d_0$, $k = 0$. Then

$$\sigma_0(A_1) = \sigma_0(A_4) = \sigma_0(A)/2 = \rho_0 \qquad \gamma_0(A_1) = \gamma_0(A_4) = b. \tag{A3.6}$$

Now consider a small perturbation $c = c_0 + \delta c$, $d = d_0 + \delta\delta$, $k \neq 0$. Denote the perturbations on the relevant quantities by

$$\begin{aligned} \sigma(A_1) &= \sigma_0(A_1) + \delta\sigma_1 \\ \sigma(A_4) &= \sigma_0(A_4) + \delta\sigma_4 \\ \sigma(A_2A_3) &= \sigma_0(A_2A_3) + \delta\sigma_{23} \\ \gamma(A_1) &= \gamma_0(A_1) + \delta\gamma_1 \\ \gamma(A_4) &= \gamma_0(A_4) + \delta\gamma_4 \end{aligned}$$

where all the $\delta\sigma$ and $\delta\gamma$ quantities depend on δc, δd and k. Note that all $\delta\sigma$ and $\delta\gamma$ are 0 if $\delta c = \delta d = k = 0$ by definition. When we also introduce $\delta\rho = \rho - \rho_0$, $\delta\sigma = \sigma(A) - \sigma_0(A)$ and $\delta\gamma = \gamma(A) - \gamma_0(A)$ we find with equation (A3.3)

$$(\delta\rho)^2 - \delta\sigma\delta\rho + \delta\gamma - \rho_0\delta\sigma = 0. \tag{A3.7}$$

With equations (A3.5) and (A3.6) we see that

$$\begin{aligned} \delta\sigma &= \delta\sigma_1 + \delta\sigma_4 \\ \delta\gamma &= \delta\gamma_1 + \delta\gamma_4 + \rho_0(\delta\sigma_1 + \delta\sigma_4) + \delta\sigma_1\delta\sigma_4 - \delta\sigma_{23}. \end{aligned} \tag{A3.8}$$

For $\delta\rho$ we thus find

$$\delta\rho = \frac{\delta\sigma_1 + \delta\sigma_4 \pm \sqrt{(\delta\sigma_1 - \delta\sigma_4)^2 + 4\delta\sigma_{23} - 4(\delta\gamma_1 + \delta\gamma_4)}}{2}. \tag{A3.9}$$

We also know that A is of the special form (6.13), because F is of type (3.1). Especially we know that A_1 and A_4 are even in k and A_2 and A_3 are odd. So all quantities in equation (A3.9) must be even in k. Moreover, $\delta\gamma_1 = O(k^2)$ because $\gamma_1 = \det(A_1) = b^{q/2}$ for all c, d if $k=0$. The same holds for $\delta\gamma_4$.

Substituting this we see that the perturbation of the stability index must be like

$$\delta\rho = \frac{\delta\sigma_1 + \delta\sigma_4 \pm \sqrt{(\delta\sigma_1 - \delta\sigma_4)^2 + k^2 U}}{2} \tag{A3.10}$$

where $\delta\sigma_1$, $\delta\sigma_4$ and U are functions of δc, δd and k^2.

Now we suppose δc, δd and k to be of the same order of magnitude, say ε. We calculate the lowest order contribution in equation (A3.10). If $k=0$ the matrix A decouples into two diagonal blocks, $A_1(\delta c)$ and $A_4(\delta d)$. Thus

$$\delta\sigma_1(\delta c, \delta d, k) = \delta\sigma_1(\delta c, 0, 0) + O(k^2) = s_1\,\delta c + O(\varepsilon^2)$$

$$\delta\sigma_4(\delta c, \delta d, k) = \delta\sigma_4(0, \delta d, 0) + O(k^2) = s_4\,\delta d + O(\varepsilon^2)$$

where $s_1 \equiv \partial\sigma_1/\partial\delta c(0,0,0)$ and $s_4 \equiv \partial\sigma_4/\partial\delta d(0,0,0)$. We remark that s_1 and s_4 are non-zero because of the assumptions on F (3.1) (monotonicity of traces for $k=0$).

If one introduces

$$U_0 \equiv U(0,0,0)$$

then

$$\delta\rho = \frac{s_1\,\delta c + s_4\,\delta d \pm \sqrt{(s_1\,\delta c - s_4\,\delta d)^2 + k^2 U_0}}{2} + O(\varepsilon^2). \tag{A3.11}$$

U_0 is typically non-zero. Its sign is related to the type of eigenvalues: a plus sign describes the unfolding of a definite pair of eigenvalues and a minus sign that of a mixed pair.

In the special case of period one (thus $A = DF$) we can do one more simplification. For then $\delta\gamma_1 = \delta\gamma_4 = 0$, because $\gamma_1 = \det(A_1) = b$ identically. So equation (A3.9) reduces to

$$\delta\rho = \frac{\delta\sigma_1 + \delta\sigma_4 \pm \sqrt{(\delta\sigma_1 - \delta\sigma_4)^2 + 4\delta\sigma_{23}}}{2}. \tag{A3.12}$$

Using the special form of equation (3.1), equation (A3.12) gives, in lowest order again, equation (A3.11), but now simply with

$$U_0 = \sigma(MN) = \mathrm{trace}(MN).$$

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